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Berry phases for the nonlocal Gross–Pitaevskii equation with a quadratic potential

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Abstract

A countable set of asymptotic space-localized solutions is constructed by the complex germ method in the adiabatic approximation for the nonstationary Gross–Pitaevskii equation with nonlocal nonlinearity and a quadratic potential. The asymptotic parameter is $1/T$, where $T \gg 1$ is the adiabatic evolution time. A generalization of the Berry phase of the linear Schrödinger equation is formulated for the Gross–Pitaevskii equation. For the solutions constructed, the Berry phases are found in explicit form.

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1. Introduction

A quantum system with slowly (adiabatically) varying parameters is characterized by the conservation of its quantum numbers throughout the time of adiabatic evolution when the system Hamiltonian has a nondegenerate spectrum. In other words, the vector of the system state remains an eigenvector at any time and gains a phase factor during the adiabatic evolution (see e.g. [1]). Berry has shown [2] that the phase gained consists of a dynamic and a geometric (topological) part. Such a division is caused by different mechanisms of their origination. The dynamic phase is associated with the evolution of the system and the topological one with the geometry of its parameter space.

Both phases result from the decomposition of the general phase of the leading term in the semiclassical approximation of the solution of the Schrödinger equation in a small parameter $1/T$, where T is the adiabatic evolution time. The dynamic phase corresponds to the zeroth-order term in the $1/T$ expansion and the Berry phase belongs to the first-order term. Inclusion of the topological phase in the general phase is necessary to determine the leading term of the semiclassical asymptotic series. The work [2] has initiated comprehensive investigations

of the geometric phases of linear quantum-mechanical equations. For details see the reviews [3–6].

It is of interest to study geometric phases for nonlinear equations describing various classes of nonlinear phenomena [7] whose nonlinear properties are due to collective interactions. An important example of systems showing nonlinear properties is the Bose–Einstein condensate (BEC) [8, 9].

In the BEC theory, various models are used. In [10, 11], the BEC model is based on the two-level Hamiltonian of a system of particles confined by a magnetic field. The system has a constant number of particles distributed between two subsystems, which are in different states. The Berry phase for this system is obtained under the condition that the system parameters responsible for the interaction of the subsystems vary slowly.

Many of the BEC models are based on the Gross–Pitaevskii equation (GPE) [8, 9, 12, 13], which is a nonstationary multidimensional cubic-nonlinear Schrödinger equation with the potential of an external field. The BEC states are described by localized solutions of the GPE. In studying these models, a number of serious mathematical problems arise. We mention two of them which are essential for our consideration.

First, no method of exact integration for the GPE is available except for the nonstationary one-dimensional Schrödinger equation that is integrable by the inverse scattering transform method [14] with no external field. In the presence of an external field, only an approximate solution can be found in terms of the soliton perturbation theory [15] with the assumption of a weak external field.

Second, even for a two-dimensional space, the localized solutions of the nonlinear Schrödinger equation (NLSE) with focusing cubic nonlinearity and no external field are unstable and eventually collapse [16], which is not observed in experiments. On the other hand, models based on the GPE with local nonlinearity can be considered as simplified versions of models with the GPE having a nonlocal nonlinearity [18]. This is a reason to consider in more detail the nonlocal operator $\int_{-\infty}^{+\infty} V(\vec{x}-\vec{y})|\Psi(\vec{y},t)|^2\,\mathrm{d}\vec{y}$ that arises in the derivation of the Gross–Pitaevskii equation. The interatomic potential $V(\vec{x}-\vec{y})$ is usually assumed to be short range, and therefore the above nonlocal nonlinear operator can be replaced by a local operator $\beta|\Psi(\vec{x},t)|^2$, where $\beta=\int_{-\infty}^{+\infty} V(\vec{y})\,\mathrm{d}\vec{y}$. More detailed analysis of the nonlocal properties can be found, for example, in [18] where the wavefunction of the one-dimensional equation is expanded in a series in $z=x-y$. The symmetry of the potential and its decrease at infinity result in an additional term $\propto \frac{\partial^2}{\partial x^2}|\psi|^2$ in the local Gross–Pitaevskii equation. The equations obtained are studied by numerical methods as the construction of an analytical solution fails even in the one-dimensional case.

In this work, we consider the one-dimensional nonlocal Gross–Pitaevskii equation and use an approach where the potential $V(x,y)$ is expanded in the variables (x,y) to the second-order terms. Exact solutions are constructed for this equation in [19, 20]. Although the interatomic quadratic potential does not decrease at infinity, the convergence of the integral is provided by a proper choice of the class of functions in which we seek solutions to the equation.

The one-dimensional GPE describes BEC states if the longitudinal dimension of the condensate is much greater than its cross dimension [18]. Unlike the NLSE with local nonlinearity, the basic properties of the solutions of the nonlocal one-dimensional GPE conserve in the multidimensional case.

The aim of this work is to obtain the Berry phase in explicit form for a one-dimensional GPE of the form

$$\{-\mathrm{i}\hbar\partial_t+\widehat{\mathcal{H}}_x(R(t),\Psi(t))\}\Psi=0, \tag{1.1}$$

$$\begin{aligned}
&\widehat{\mathcal{H}}_{\varkappa}(R(t),\Psi(t)) = \widehat{\mathcal{H}}(R(t)) + \varkappa\widehat{V}(R(t),\Psi(t)),\\
&\widehat{\mathcal{H}}(R(t)) = \frac{\mu(t)\hat{p}^2}{2} + \frac{\sigma(t)x^2}{2} + \frac{\rho(t)(x\hat{p}+\hat{p}x)}{2},\\
&\widehat{V}(R(t),\Psi(t)) = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}y[a(t)x^2 + 2b(t)xy + c(t)y^2]|\Psi(y,t)|^2,
\end{aligned} \tag{1.2}$$

which involves a quadratic potential and an external field of the harmonic oscillator type. Here $a(t)$, $b(t)$ and $c(t)$ are the potential parameters, $\varkappa$ is the nonlinearity parameter, and $\mu(t)$, $\sigma(t)$ and $\rho(t)$ are time-dependent system parameters. Therefore, the Hamiltonian depends on time via the set of parameters $R(t) = (\mu(t), \sigma(t), \rho(t), a(t), b(t)$ and $c(t))$. The quadratic potential in equations (1.1) and (1.2) models the magnetic traps that confine the condensate [17].

The problem of constructing solutions for equation (1.1) with an external potential is solved with the use of the semiclassical integration method developed in [19–22], where equation (1.1) is called the 'Hartree-type equation'. Below we call the nonlinear operator $\widehat{\mathcal{H}}_{\varkappa}$ in equation (1.1) the nonlinear Hamiltonian. The semiclassical integration method allows one to construct localized solutions, approximate for the potential of general form and exact for the quadratic potential. We use these solutions to find and study geometric Berry phases.

Let us give necessary facts from the theory of geometric phases of linear quantum-mechanical systems to find Berry phase in the nonlinear case. Following [2] (see also [5, 3]), we consider a Hamiltonian $\widehat{\mathcal{H}}(t) = \widehat{\mathcal{H}}(R(t))$, which depends on time t via a set of slowly varying T-periodic functions $R(t)$. Denote by $\Psi_{E_n(R(t))}(x, R(t))$ the eigenfunctions of the instantaneous Hamiltonian $\widehat{\mathcal{H}}(R(t))$:

$$\widehat{\mathcal{H}}(R(t))\Psi_{E_n(R(t))}(x,R(t)) = E_n(R(t))\Psi_{E_n(R(t))}(x,R(t)). \tag{1.3}$$

Assume the spectrum of the Hamiltonian $\widehat{\mathcal{H}}(R(t))$ to be nondegenerate at any fixed time and set the Cauchy problem

$$\{-\mathrm{i}\hbar\partial_t + \widehat{\mathcal{H}}(R(t))\}\Psi = 0, \tag{1.4}$$

$$\Psi|_{t=t_0} = \Psi_{E_n(R(t_0))}(x, R(t_0)). \tag{1.5}$$

According to the adiabatic theorem [1][3], the quantum numbers of the system will conserve throughout the adiabatic evolution time T if the parameters $R(t)$ depend on time adiabatically. In other words, a solution of the Cauchy problem (1.4), (1.5) differs from the initial state by the phase factor

$$\Psi(x,T) = \exp[\mathrm{i}\phi_n(T)]\Psi_{E_n(R(T))}(x,R(T)). \tag{1.7}$$

Following [2], represent the phase $\phi_n(T)$ as

$$\phi_n(T) = \delta_n(T) + \gamma_n(T), \tag{1.8}$$

where $\delta_n(T)$ is the dynamic phase given by

$$\delta_n(T) = -\frac{1}{\hbar}\int_0^T E_n(R(t))\,\mathrm{d}t. \tag{1.9}$$

[3] It is generally agreed that a system evolves adiabatically if

$$\max_{i=\overline{1,n}} \dot{R}_i \frac{T}{R_i} \ll 1, \tag{1.6}$$

where R_i are the parameters of the Hamiltonian (see [23]).

The phase $\gamma_n(T)$ is called an *adiabatic Berry phase*, and for a linear Schrödinger equation it is determined by the expression

$$\dot{\gamma}_n(t) = \mathrm{i}\langle \Psi_{E_n(R(t))}(x, R(t)) \mid \dot{\Psi}_{E_n(R(t))}(x, R(t))\rangle \tag{1.10}$$

or

$$\gamma_n(T) = \mathrm{i}\int_0^T \langle \Psi_{E_n(R(t))}(x, R(t)) \mid \dot{\Psi}_{E_n(R(t))}(x, R(t))\rangle \,\mathrm{d}t = \oint_C A_j^n \,\mathrm{d}R^j. \tag{1.11}$$

Here

$$A_j^n = \mathrm{i}\left\langle \Psi_n \left| \frac{\partial \Psi_n}{\partial R^j} \right\rangle \right. \tag{1.12}$$

and C is a closed contour in the parameter space. The functions A_j^n act as components of an 'induced gauge field' since under the gauge transformations

$$\Psi_n \to \exp(\mathrm{i}\xi_n(R))\Psi_n \tag{1.13}$$

the quantities A^n are transformed as

$$A_j^n \to A_j^n + \frac{\partial \xi_n}{\partial R^j}. \tag{1.14}$$

Expression (1.11) does not depend on the transformation (1.14) as the contour C is closed. The dynamic phase $\delta_n(T)$ characterizes the mean value of the system energy, and the geometric phase $\gamma_n(T)$ does not depend on the system dynamics. Berry's phase depends on the geometry of the system parameter space and on the type of the contour C.

Assume the adiabatic theorem to hold for nonlinear equations and define Berry's phase by relations (1.8) and (1.9). Formula (1.11), being equivalent to (1.8) in the linear case, needs additional substantiation for nonlinear equations.

In this work, we use the approach developed in [24, 25] for linear equations to find Berry's phase. The approach is based on the exact (or approximate) solution of the Cauchy problem (1.4), (1.5) which is expanded in an adiabatic parameter. For the linear equation (1.1), the exact solution of the Cauchy problem is constructed using the method proposed in [19]. Here, the initial Hamiltonian in the initial condition of the Cauchy problem (1.5) is replaced by (1.2) and the solution is determined by two auxiliary systems of ordinary differential equations: the Hamilton–Ehrenfest system and the system in variations. Solutions of these systems are unknown when the coefficients are arbitrary functions of time. When the coefficients depend on time adiabatically, we can seek the solution in the form of an expansion in the adiabaticity parameter which is taken as $1/T$, where T is a 'long' characteristic time, for example, the adiabatic evolution period of the system.

If the Hamilton–Ehrenfest system and the system in variations are solved accurate to $O(1/T)$, the equation considered is solved with the same accuracy, and thus we obtain a solution of equation (1.1) in the adiabatic approximation. For such a solution, Berry's phase can be found in explicit form.

2. The Hamilton–Ehrenfest system

For a linear operator $\widehat{A}$ we define its mean value in a state $\Psi(t)$ as

$$\langle \widehat{A}(t)\rangle = \frac{1}{\|\Psi(t)\|^2}\langle \Psi(t)|\widehat{A}|\Psi(t)\rangle = A_\Psi(t, \hbar). \tag{2.1}$$

On the solutions $\Psi(t)$ of equation (1.4) we have

$$\frac{\mathrm{d}\langle \widehat{A}(t)\rangle}{\mathrm{d}t} = \left\langle \frac{\partial \hat{A}(t)}{\partial t} \right\rangle + \frac{\mathrm{i}}{\hbar}\langle [\widehat{\mathcal{H}}_x(t, \Psi(t)), \widehat{A}(t)]\rangle, \tag{2.2}$$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator of linear operators $\hat{A}$ and $\hat{B}$.

Similar to the linear case, we call (2.2) the Ehrenfest equation. From this equation with $\widehat{A} = 1$ it follows, in particular, that the norm of a solution of equation (1.4) is conserved, that is,

$$\|\Psi(x,t)\|^2 = \|\Psi(x,0)\|^2 = \|\Psi\|^2.$$

So it is convenient to use the nonlinear parameter $\tilde{\varkappa} = \varkappa\|\Psi\|^2$ instead of $\varkappa$.

Denote by

$$\alpha_\Psi^{(l,k)}(t,\hbar) = \frac{1}{\|\Psi\|^2}\int_{-\infty}^{+\infty} \Psi^*(y,t)\{(\Delta\hat{p}_y)^l(\Delta y)^k\}\Psi(y,t)\,\mathrm{d}y, \qquad k,l = \overline{0,\infty},$$

the moments of the $(k+l)$th order centred relative to $x_\Psi(t,\hbar)$ and $p_\Psi(t,\hbar)$. Here $\Delta\hat{p}_y = -\mathrm{i}\hbar\partial_y - p_\Psi(t,\hbar)$, and $\{(\Delta\hat{p}_y)^l(\Delta y)^k\}$ is a Weyl-ordered operator with the symbol $(\Delta p_y)^l(\Delta y)^k$. In particular,

$$\sigma_{xx}(t,\hbar) = \alpha_\Psi^{(0,2)}(t,\hbar), \qquad \sigma_{pp}(t,\hbar) = \alpha_\Psi^{(2,0)}(t,\hbar), \qquad \sigma_{xp}(t,\hbar) = \alpha_\Psi^{(1,1)}(t,\hbar)$$

are the variances of the coordinates and momenta and of the correlation function of the coordinates and momenta, respectively.

Consider the first-order operators $\hat{p}$ and x and the centred second-order operators $(\Delta x)^2$, $(\Delta\hat{p})^2$, $(\Delta x\Delta\hat{p} + \Delta\hat{p}\Delta x)/2$, where $\Delta\hat{p} = \hat{p} - p_\Psi(t,\hbar)$, $\Delta x = x - x_\Psi(t,\hbar)$, $p_\Psi(t,\hbar) = \langle\hat{p}\rangle$ and $x_\Psi(t,\hbar) = \langle x\rangle$.

The Ehrenfest system for the mean values of these operators reads

$$\begin{cases} \dot{p} = -\sigma_0(t)x - \rho(t)p, \\ \dot{x} = \mu(t)p + \rho(t)x, \\ \dot{\sigma}_{xx} = 2\mu(t)\sigma_{xp} + 2\rho(t)\sigma_{xx}, \\ \dot{\sigma}_{xp} = \mu(t)\sigma_{pp} - \tilde{\sigma}(t)\sigma_{xx}, \\ \dot{\sigma}_{pp} = -2\rho(t)\sigma_{pp} - 2\tilde{\sigma}(t)\sigma_{xp}. \end{cases} \tag{2.3}$$

Here

$$\sigma_0(t) = \sigma(t) + \tilde{\varkappa}(a(t) + b(t)), \qquad \tilde{\sigma}(t) = \sigma(t) + \tilde{\varkappa}a(t).$$

For system (2.3), let us set a Cauchy problem with initial conditions

$$\begin{aligned} &p|_{t=s} = p_0, && x|_{t=s} = x_0, \\ &\sigma_{pp}|_{t=s} = \sigma_{0pp}, && \sigma_{xp}|_{t=s} = \sigma_{0xp}, && \sigma_{xx}|_{t=s} = \sigma_{0xx}. \end{aligned} \tag{2.4}$$

We call (2.3) *the second-order Hamilton–Ehrenfest system* (HES) related to equation (1.4).

Consider the HES (2.3) as a dynamical system which is not related to equation (1.4). Apparently, not all solutions of the HES can be obtained as mean values of the corresponding operators on the solutions of equation (1.4). For example, the mean values must satisfy the Schrödinger uncertainty relation

$$\sigma_{pp}\sigma_{xx} - \sigma_{xp}^2 \geqslant \frac{\hbar^2}{4} \tag{2.5}$$

for the second-order moments (for the higher order relations see [26]). It can readily be seen that the HES admits the trivial solution $p = 0, x = 0, \alpha^{(k,l)} = 0, k+l = 2$. The left-hand side of (2.5) is the integral of motion of the HES (2.3) (see [27]). Hence, it suffices that the uncertainty relation be fulfilled at the initial time. The uncertainty relations will be fulfilled automatically if the initial conditions for (2.3) are taken as

$$\begin{aligned} &p|_{t=s} = p_0 = p_\psi(\hbar), && x|_{t=s} = x_0 = x_\psi(\hbar), \\ &\sigma_{pp}|_{t=s} = \alpha_\psi^{(2,0)}(\hbar), && \sigma_{xp}|_{t=s} = \alpha_\psi^{(1,1)}(\hbar), && \sigma_{xx}|_{t=s} = \alpha_\psi^{(0,2)}(\hbar), \end{aligned} \tag{2.6}$$

where $\psi(x,\hbar)$ is the initial condition for equation (1.4):

$$\Psi(x,\hbar,t)|_{t=0} = \psi(x,\hbar). \tag{2.7}$$

Denote a trajectory in an extended phase space by $\mathfrak{g} = \mathfrak{g}(t,\mathfrak{C}) \in \mathbb{R}^5$, where

$$\begin{aligned} &\mathfrak{g}(t,\mathfrak{C}) = (P(t,\mathfrak{C}), X(t,\mathfrak{C}), \sigma_{pp}(t,\mathfrak{C}), \sigma_{px}(t,\mathfrak{C}), \sigma_{xx}(t,\mathfrak{C}))^{\mathsf{T}},\\ &\mathfrak{C} = (C_1, C_2, C_3, C_4, C_5)^{\mathsf{T}}, \end{aligned} \tag{2.8}$$

is the general solution of the Hamilton–Ehrenfest system (2.3) and $\hat{\mathfrak{g}}$ is the operator column

$$\hat{\mathfrak{g}} = \left(\hat{p}, x, (\Delta\hat{p})^2, \tfrac{1}{2}(\Delta\hat{p}\Delta x - \Delta x\Delta\hat{p}), (\Delta x)^2\right)^{\mathsf{T}}. \tag{2.9}$$

Here $C_l, l = \overline{1,5}$, are arbitrary constants which can be expressed in terms of the initial conditions (2.4). The matrix B^{T} is transposed to the matrix B. The system (2.3) can be rewritten in the form

$$\dot{\mathfrak{g}} = \mathfrak{A}\mathfrak{g}, \qquad \mathfrak{g}|_{t=s} = \mathfrak{g}_0, \tag{2.10}$$

where

$$\mathfrak{A} = \begin{pmatrix} -\rho(t) & -\sigma_0(t) & 0 & 0 & 0\\ \mu(t) & \rho(t) & 0 & 0 & 0\\ 0 & 0 & -2\rho(t) & -2\tilde{\sigma}(t) & 0\\ 0 & 0 & \mu(t) & 0 & -\tilde{\sigma}(t)\\ 0 & 0 & 0 & 2\mu(t) & 2\rho(t) \end{pmatrix}.$$

3. The associated linear Schrödinger equation

Let us seek a solution to equation (1.1) in the form of the ansatz

$$\Psi(x,t,\hbar) = \varphi\left(\frac{\Delta x}{\sqrt{\hbar}}, t, \sqrt{\hbar}\right)\exp\left[\frac{\mathrm{i}}{\hbar}(S(t,\mathfrak{C}) + P(t,\mathfrak{C})\Delta x)\right]. \tag{3.1}$$

Here the function $\varphi(\xi,t,\sqrt{\hbar})$ belongs to the Schwartz space $\mathbb{S}$ in the variable $\xi = \Delta x/\sqrt{\hbar}$ and depends regularly on $\sqrt{\hbar}$; $\Delta x = x - X(t,\mathfrak{C})$. The real functions $S(t,\mathfrak{C})$ and $Z(t,\mathfrak{C}) = (P(t,\mathfrak{C}), X(t,\mathfrak{C}))$ that characterize the solution are to be determined.

Expand the operators in equation (1.1) in a Taylor series in $\Delta x = x - x_\Psi(t,\hbar)$, $\Delta y = y - x_\Psi(t,\hbar)$, $\Delta\hat{p} = \hat{p} - p_\Psi(t,\hbar)$, and $\Delta\hat{z} = \hat{z} - z_\Psi(t,\hbar)$. Then equation (1.1) takes the form

$$\begin{aligned} &\left\{-\mathrm{i}\hbar\partial_t + \mathfrak{H}(t,\Psi) + \langle\mathfrak{H}_z(t,\Psi), \Delta\hat{z}\rangle + \frac{1}{2}\langle\Delta\hat{z}, \mathfrak{H}_{zz}(t,\Psi)\Delta\hat{z}\rangle\right\}\Psi = 0,\\ &\mathfrak{H}(t,\Psi) = \frac{\mu(t)p_\Psi^2(t,\hbar)}{2} + \frac{\sigma(t)x_\Psi^2(t,\hbar)}{2} + \rho(t)x_\Psi(t,\hbar)p_\Psi(t,\hbar)\\ &\qquad + \frac{\tilde{\varkappa}}{2}c\alpha_\Psi^{(0,2)}(t,\hbar) + \frac{\tilde{\varkappa}}{2}(a+2b+c)x_\Psi^2(t,\hbar),\\ &\mathfrak{H}_z(t,\Psi) = \begin{pmatrix} \mu(t)p_\Psi(t,\hbar) + \rho(t)x_\Psi(t,\hbar)\\ \sigma(t)x_\Psi(t,\hbar) + \rho(t)p_\Psi(t,\hbar) + \tilde{\varkappa}(a+b)x_\Psi(t,\hbar)\end{pmatrix},\\ &\mathfrak{H}_{zz}(t,\Psi) = \begin{pmatrix} \mu(t) & \rho(t)\\ \rho(t) & \widetilde{\sigma}(t)\end{pmatrix}. \end{aligned} \tag{3.2}$$

Let us associate the nonlinear equation (3.2) with the linear equation that is obtained from (3.2) by formal substitution of the solution of the HES (2.3) instead of the corresponding

mean values of the coordinate and momenta operators and second-order centred moments. The resulting linear equation is

$$\left\{-\mathrm{i}\hbar\partial_t + \mathfrak{H}(t,\mathfrak{C}) + \langle \mathfrak{H}_z(t,\mathfrak{C}),\Delta\hat{z}\rangle + \frac{1}{2}\langle \Delta\hat{z}, \mathfrak{H}_{zz}(t,\mathfrak{C})\Delta\hat{z}\rangle\right\}\Phi = 0,$$

$$\begin{aligned}\mathfrak{H}(t,\mathfrak{C}) &= \frac{\mu(t)P^2(t,\mathfrak{C})}{2} + \frac{\sigma(t)X^2(t,\mathfrak{C})}{2} + \rho(t)X(t,\mathfrak{C})P(t,\mathfrak{C})\\ &\quad + \frac{\tilde{\varkappa}}{2}c\sigma(t)_{xx}(t,\mathfrak{C},\hbar) + \frac{\tilde{\varkappa}}{2}(a+2b+c)X^2(t,\mathfrak{C}),\\ \mathfrak{H}_z(t,\mathfrak{C}) &= \begin{pmatrix}\mu(t)P(t,\mathfrak{C}) + \rho(t)X(t,\mathfrak{C})\\ \sigma_0(t)X(t,\mathfrak{C}) + \rho(t)P(t,\mathfrak{C})\end{pmatrix},\\ \mathfrak{H}_{zz}(t,\mathfrak{C}) &= \begin{pmatrix}\mu(t) & \rho(t)\\ \rho(t) & \widetilde{\sigma}(t)\end{pmatrix}.\end{aligned} \tag{3.3}$$

We call equation (3.3) the *associated linear Schrödinger equation*.

By direct check we see that the function

$$\Phi_0(x,t,\mathfrak{C}) = |0,t,\mathfrak{C}\rangle = N_\hbar\left(\frac{C(0)}{C(t)}\right)^{1/2}\exp\left\{\frac{\mathrm{i}}{\hbar}\left(S(t,\mathfrak{C}) + P(t,\mathfrak{C})\Delta x + \frac{1}{2}\frac{B(t)}{C(t)}\Delta x^2\right)\right\} \tag{3.4}$$

is a solution of equation (3.3). Here

$$S(t,\mathfrak{C}) = \int_0^t (P(t,\mathfrak{C})\dot{X}(t,\mathfrak{C}) - \mathfrak{H}(t,\mathfrak{C}))\,\mathrm{d}t, \tag{3.5}$$

and $B(t)$ and $C(t)$ denote, respectively, the momentum and the coordinate part of the solution

$$a(t) = \begin{pmatrix}B(t)\\ C(t)\end{pmatrix} \tag{3.6}$$

of the system in variations

$$\dot{a} = J\mathfrak{H}_{zz}(t)a, \qquad a|_{t=s} = a_0, \tag{3.7}$$

related to equation (3.3).

The normalizing condition $\|\Psi\|^2 = 1$ yields $N_\hbar = (\pi\hbar)^{-1/4}(|C(0)|)^{-1/2}$.

Let us introduce the notation

$$\hat{a}(t) = N_a(C(t)\Delta\hat{p} - B(t)\Delta x).$$

If $C(t)$ and $B(t)$ are solutions of equations (3.7), then the operator $\hat{a}(t)$ commutes with the operator of the associated equation (3.3). Therefore, the function

$$\Phi_n(x,t,\mathfrak{C}) = \frac{1}{\sqrt{n!}}(\hat{a}^+(t))^n\Phi_0(x,t,\mathfrak{C}), \qquad n = \overline{0,\infty},$$

will be a solution of the Schrödinger equation (3.3). Commutating the operators $\hat{a}^+(t)$ with the operator of multiplication by the function $\Phi_0(x,t,\mathfrak{C}) = |0,t,\mathfrak{C}\rangle$, we obtain the following representation for the Fock basis of solutions of the linear equation (3.3):

$$\begin{aligned}\Phi_n(x,t,\mathfrak{C}) &= \frac{(\mathrm{i})^n}{\sqrt{n!}}(N_a^*)^n\Phi_0(x,t,\mathfrak{C})\left[\frac{|C(t)|\sqrt{\hbar}}{C(t)}\right]^n\left[\sqrt{\hbar}|C(t)|\frac{\partial}{\partial x} - \frac{2}{\sqrt{\hbar}|C(t)|}\Delta x\right]^n 1\\ &= \frac{1}{\sqrt{n!}}(N_a^*)^n\Phi_0(x,t,\mathfrak{C})\left(\frac{\mathrm{i}}{\sqrt{2}}\right)^n(\sqrt{\hbar})^n\exp\left(-\mathrm{i}n\operatorname{Arg}C(t)\right)H_n\left(\Delta x\sqrt{\frac{\operatorname{Im}Q(t)}{\hbar}}\right),\end{aligned}$$

where $H_n(\xi)$ are the Hermitian polynomials and $Q(t) = B(t)C^{-1}(t)$. Finding $N_a = (1/\sqrt{2\hbar})\exp[-\mathrm{i}\,\mathrm{Arg}\,C(0)]$ from the condition $[\hat{a}(t), \hat{a}^+(t)] = 1$, we have

$$\begin{aligned}\Phi_n(x,t,\mathfrak{C}) = |n,t,\mathfrak{C}\rangle &= \frac{1}{n!}[\hat{a^+}(t)]^n|0,t,\mathfrak{C}\rangle \\ &= \frac{1}{\sqrt{n!}}|0,t\rangle\left(\frac{\mathrm{i}}{\sqrt{2}}\right)^n \exp[-\mathrm{i}n(\mathrm{Arg}\,C(t) - \mathrm{Arg}\,C(0))]H_n(\xi), \qquad (3.8)\end{aligned}$$

$$\xi = \sqrt{\frac{\mathrm{Im}\,Q(t)}{\hbar}}\Delta x.$$

Using the properties of Hermitian polynomials, we obtain the mean values of the momentum and coordinate operators and the corresponding variances:

$$\begin{aligned} &x_{\Phi_n} = 0, \qquad p_{\Phi_n} = 0, \\ &\alpha^{(2,0)}_{\Phi_n} = \sigma_{pp}(t,\hbar) = \hbar\frac{\widetilde{\sigma}(t)(2n+1)}{2\mu(t)\,\mathrm{Im}Q(t)}, \\ &\alpha^{(1,1)}_{\Phi_n} = \sigma_{xp}(t,\hbar) = -\hbar\frac{\rho(t)(2n+1)}{2\mu(t)\,\mathrm{Im}Q(t)}, \\ &\alpha^{(0,2)}_{\Phi_n}(t,\hbar) = \sigma_{xx}(t,\hbar) = \hbar\frac{(2n+1)}{2\mathrm{Im}Q(t)}. \end{aligned} \qquad (3.9)$$

The functions $\Phi_n(x,t,\mathfrak{C})$ are solutions of equation (1.1) for properly chosen $\mathfrak{C}$, such that the solutions of the Hamilton–Ehrenfest system solutions (2.3) coincide with equations (3.9). Denoting this set of parameters by $\overline{\mathfrak{C}}_n$, we obtain

$$\Psi_n(x,t) = \Phi_n(x,t,\overline{\mathfrak{C}}_n). \qquad (3.10)$$

The subscript n in $\overline{\mathfrak{C}}_n$ implies that every function $\Psi_n(x,t)$ has its own set of parameters $\overline{\mathfrak{C}}_n$.

4. Eigenfunctions of the instantaneous nonlinear Hamiltonian

To construct the Berry phase using formulae (1.7) and (1.8), we have to solve a spectral problem for the instantaneous Hamiltonian (1.2) in the class of functions (3.1)

$$\widehat{\mathcal{H}}_x(R,\psi_n)\psi_n = E_n\psi_n. \qquad (4.1)$$

To solve this problem, consider the nonstationary Schrödinger equation

$$\{-\mathrm{i}\hbar\partial_t + \hat{\mathcal{H}}_x(R,\Psi(t))\}\Psi = 0. \qquad (4.2)$$

Solutions of equation (4.2) of the form

$$\Psi(x,t) = \exp\left\{-\frac{\mathrm{i}}{\hbar}E_n(R)t\right\}\psi_n(x,R) \qquad (4.3)$$

provide a solution of the spectral problem (4.1) where $\psi_n(x,R)$ and $E_n(R)$ are the instantaneous eigenfunctions and eigenvalues of the Hamiltonian $\widehat{\mathcal{H}}_x(R,\psi_n(R))$, respectively.

The Hamilton–Ehrenfest system (2.3) for the first-order moments related to equation (4.2) takes the form

$$\begin{cases}\dot{p} = -\sigma_0 x - \rho p, \\ \dot{x} = \mu p + \rho x, \end{cases} \qquad (4.4)$$

and for the second-order moments we have

$$\begin{cases}\dot{\sigma}_{xx} = 2\mu\sigma_{xp} + 2\rho\sigma_{xx}, \\ \dot{\sigma}_{xp} = \mu\sigma_{pp} - \tilde{\sigma}\sigma_{xx}, \\ \dot{\sigma}_{pp} = -2\rho\sigma_{pp} - 2\tilde{\sigma}\sigma_{xp}, \end{cases} \qquad (4.5)$$

where

$$\sigma_0 = \sigma + \tilde{\varkappa}(a+b), \qquad \tilde{\sigma} = \sigma + \tilde{\varkappa}a.$$

Let us denote

$$\tilde{\Omega} = \sqrt{\sigma_0\mu - \rho^2}, \qquad \Omega = \sqrt{\tilde{\sigma}\mu - \rho^2}. \tag{4.6}$$

The spectral problem is associated only with the time-localized solutions of the system (4.4), (4.5) which are stable in the linear approximation. The localization condition holds when

$$\tilde{\Omega}^2 = \sigma_0\mu - \rho^2 > 0, \qquad \Omega^2 = \tilde{\sigma}\mu - \rho^2 > 0.$$

In this case, the general solution of the system (4.4) is given by

$$\begin{aligned} X(t) &= C_1 \sin\tilde{\Omega}t + C_2\cos\tilde{\Omega}t, \\ P(t) &= \frac{1}{\mu}(\tilde{\Omega}C_1 - \rho C_2)\cos\tilde{\Omega}t - \frac{1}{\mu}(\tilde{\Omega}C_2 + \rho C_1)\sin\tilde{\Omega}t. \end{aligned} \tag{4.7}$$

Accordingly, for the system (4.5) we have

$$\begin{aligned} \sigma_{xx}(t) &= C_3\sin 2\Omega t + C_4\cos 2\Omega t + C_5, \\ \sigma_{xp}(t) &= \frac{1}{\mu}(\Omega C_3 - \rho C_4)\cos 2\Omega t - \frac{1}{\mu}(\Omega C_4 + \rho C_3)\sin 2\Omega t - \frac{\rho}{\mu}C_5, \\ \sigma_{pp}(t) &= \frac{1}{\mu^2}((\rho^2-\Omega^2)C_3 + 2\rho\Omega C_4)\sin 2\Omega t + \frac{1}{\mu^2}((\rho^2-\Omega^2)C_4 - 2\rho\Omega C_3)\cos 2\Omega t + \frac{\tilde{\sigma}}{\mu}C_5. \end{aligned} \tag{4.8}$$

Here C_l, $l = \overline{1,5}$, are arbitrary constants.

Following equation (2.8), denote the general solution of the Hamilton–Ehrenfest system (4.4), (4.5) by $\mathfrak{g} = \mathfrak{g}(t, \mathfrak{C}) \in \mathbb{R}^5$. To solve the spectral problem, we need for the stationary solution of the Hamilton–Ehrenfest system (4.4), (4.5) $\mathfrak{g} = \mathfrak{g}(t, \mathfrak{C}_s) \in \mathbb{R}^5$ that is obtained with the parameters taken as follows:

$$\mathfrak{C}_s = (0, 0, 0, 0, C_5)^{\mathsf{T}}. \tag{4.9}$$

The system in variations (3.7) for the corresponding associated linear equation (3.3) becomes

$$\dot{a}(t) = \begin{pmatrix} -\rho & -\widetilde{\sigma} \\ \mu & \rho \end{pmatrix} a(t). \tag{4.10}$$

Let us set a Floquet problem [28] for the system in variations (4.10):

$$a(t+T) = \mathrm{e}^{\mathrm{i}\Omega T}a(t). \tag{4.11}$$

The quasiperiodicity condition (4.11) for solutions of the system in variations (4.10) is sufficient for the solutions of the Hamilton–Ehrenfest system in the linear approximation to be stable. A solution of the Floquet problem (4.10), (4.11), with the normalization condition

$$\{a(t), a^*(t)\} = 2\mathrm{i}, \qquad \{a_1, a_2\} = \langle a_1, J^t a_2\rangle,$$

can be written in the form

$$a(t) = \frac{\mathrm{e}^{\mathrm{i}\Omega t}}{\sqrt{\Omega\mu}}\begin{pmatrix} -\rho + \mathrm{i}\Omega \\ \mu \end{pmatrix}. \tag{4.12}$$

Let us seek solutions to equation (4.2) in the form (3.1). Then the solution (3.4) of the corresponding associated equation (3.3) is given by

$$\Phi_0(x,t,\mathfrak{C}) = \sqrt[4]{\frac{1}{\pi\hbar}}\sqrt[2]{\frac{1}{|C(t)|}}\exp\left\{\frac{\mathrm{i}}{\hbar}\left(-\frac{\tilde{\varkappa}}{2}cC_5t - \frac{1}{2}\hbar\Omega t + \frac{1}{2}\frac{B(t)}{C(t)}\Delta x^2\right)\right\}. \tag{4.13}$$

Accordingly, for the Fock basis (3.8) we obtain

$$\Phi_n(x,t,\mathfrak{C}) = \frac{\mathrm{i}^n}{\sqrt{n!}}\exp\{-\mathrm{i}n\Omega t\}\left(\frac{1}{\sqrt{2}}\right)^n H_n\left(\sqrt{\frac{\Omega}{\hbar\mu}}\Delta x\right)\Phi_0(x,t,\mathfrak{C}). \quad (4.14)$$

Taking the parameters of the solution of the Hamilton–Ehrenfest system in the form

$$\overline{\mathfrak{C}}_n = (C_1, C_2, C_3, C_4, C_5)^{\mathsf{T}} = \left(0,0,0,0,\hbar\frac{\mu(2n+1)}{2\Omega}\right)^{\mathsf{T}}, \quad (4.15)$$

we obtain that solutions of the associated equation (3.3) will be solutions of the original equation (4.2):

$$\Psi_n(x,t) = \Phi_n(x,t,\overline{\mathfrak{C}}_n) = \frac{\mathrm{i}^n}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}}\right)^n\left(\frac{1}{\pi\hbar}\right)^{1/4}\left(\frac{\Omega}{\mu}\right)^{1/4}\exp\left\{\frac{\mathrm{i}}{\hbar}\left(-\left(n+\frac{1}{2}\right)\hbar\frac{\tilde{\varkappa}c\mu}{2\Omega}t\right.\right.$$
$$\left.\left.-\left(n+\frac{1}{2}\right)\hbar\Omega t - \frac{1}{2}\frac{\rho}{\mu}x^2\right) - \frac{1}{2\hbar}\frac{\Omega}{\mu}x^2\right\} H_n\left(\sqrt{\frac{\Omega}{\hbar\mu}}x\right). \quad (4.16)$$

Hence, the eigenfunctions of the operator (1.2) have the form

$$\psi_n(x,R) = \frac{\mathrm{i}^n}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}}\right)^n\left(\frac{1}{\pi\hbar}\right)^{1/4}\left(\frac{\Omega}{\mu}\right)^{1/4}\exp\left\{-\frac{\mathrm{i}}{2\hbar}\frac{\rho}{\mu}x^2 - \frac{1}{2\hbar}\frac{\Omega}{\mu}x^2\right\} H_n\left(\sqrt{\frac{\Omega}{\hbar\mu}}x\right), \quad (4.17)$$

and the corresponding eigenvalues are given by

$$E_n = \hbar\left(n+\frac{1}{2}\right)\left(\frac{\tilde{\varkappa}c\mu}{2\Omega}+\Omega\right). \quad (4.18)$$

5. Adiabatic approximation

Assume that the evolution of a system is adiabatic. To this case (see (1.6)) the parameters of the Hamiltonian slowly vary in time, and we can introduce, along with the 'fast' time t appearing in the time derivative, a 'slow' time s on which the parameters of the Hamiltonian ($R(t)$=$R(s)$) depend.

Let the 'fast' and the 'slow' times be related as

$$s = \frac{t}{T}, \quad (5.1)$$

where T is the evolution period of the system.

For equation (1.1), we consider the Cauchy problem

$$\{-\mathrm{i}\hbar\partial_t + \widehat{\mathcal{H}}_x(R(s),\Psi(t))\}\Psi = 0, \quad (5.2)$$

$$\Psi(x,t)|_{t=0} = \psi_n(x,R(0)), \quad (5.3)$$

where $\psi_n(x,R(0))$ are the eigenfunctions of the instantaneous Hamiltonian $\widehat{\mathcal{H}}_x(R(0),\Psi(0))$.

As noted above, to solve equation (5.2) in adiabatic approximation we have to solve the Hamilton–Ehrenfest system and the system in variations accurate to the first order of $1/T$.

The Hamilton–Ehrenfest system for the first-order moments can be written as

$$\begin{cases} \dfrac{1}{T}p' = -\sigma_0(s)x - \rho(s)p, \\ \dfrac{1}{T}x' = \mu(s)p + \rho(s)x, \end{cases} \quad (5.4)$$

and for the second-order moments we have

$$\begin{cases} \dfrac{1}{T}\sigma'_{xx} = 2\mu(s)\sigma_{xp} + 2\rho(s)\sigma_{xx}, \\ \dfrac{1}{T}\sigma'_{xp} = \mu(s)\sigma_{pp} - \widetilde{\sigma}(s)\sigma_{xx}, \\ \dfrac{1}{T}\sigma'_{pp} = -2\rho(s)\sigma_{pp} - 2\tilde{\sigma}(s)\sigma_{xp}, \end{cases} \tag{5.5}$$

where $a' = \mathrm{d}a/\mathrm{d}s$.

Let us seek a solution to the Hamilton–Ehrenfest system (5.4), (5.5) in the form

$$\begin{aligned} &x(t) = x^{(0)}(s) + \frac{1}{T}x^{(1)}(s), \qquad p(t) = p^{(0)}(s) + \frac{1}{T}p^{(1)}(s), \\ &\Sigma(t) = \Sigma^{(0)}(s) + \frac{1}{T}\Sigma^{(1)}(s), \end{aligned} \tag{5.6}$$

where

$$\Sigma = \begin{pmatrix} \sigma_{xx} \\ \sigma_{xp} \\ \sigma_{pp} \end{pmatrix}. \tag{5.7}$$

Substituting (5.6) in the Hamilton–Ehrenfest system (5.4), (5.5) and equating the term of the same powers in $1/T$, we obtain

$$x^{(0)} = x^{(1)} = 0, \qquad p^{(0)} = p^{(1)} = 0,$$

$$\Sigma^{(0)}(s) = C_1 \begin{pmatrix} \frac{\mu(s)}{\Omega(s)} \\ -\frac{\rho(s)}{\Omega(s)} \\ \frac{\widetilde{\sigma}(s)}{\Omega(s)} \end{pmatrix}, \qquad \Sigma^{(1)}(s) = \sigma^{(1)}_{xx}(s) \begin{pmatrix} 1 \\ -\frac{\rho(s)}{\mu(s)} \\ \frac{\widetilde{\sigma}(s)}{\mu(s)} \end{pmatrix} + C_1 \begin{pmatrix} 0 \\ \frac{1}{2\rho(s)}\left(\frac{\mu(s)}{\Omega(s)}\right)' \\ -\frac{1}{\mu(s)}\left(\frac{\rho(s)}{\Omega(s)}\right)' \end{pmatrix}. \tag{5.8}$$

The function $\sigma^{(1)}_{xx}(s)$ is determined by the condition of existence of the following approximation and has the form

$$\sigma^{(1)}_{xx}(s) = C_1 \frac{\mu^2(s)}{2\Omega^3(s)}\left(\frac{\rho(s)}{\mu(s)}\right)' + C_2\left(\frac{\mu(s)}{\Omega(s)}\right). \tag{5.9}$$

A trajectory in the extended phase space $\mathfrak{g} = \mathfrak{g}(t, \mathfrak{C}) \in \mathbb{R}^5$ has the form

$$\mathfrak{g}(t, \mathfrak{C}) = (0, 0, \sigma_{pp}(t, \mathfrak{C}), \sigma_{px}(t, \mathfrak{C}), \sigma_{xx}(t, \mathfrak{C}))^{\intercal}, \qquad \mathfrak{C} = (C_1, C_2)^{\intercal}. \tag{5.10}$$

Applying the change of variables (5.1) to the system in variations (3.7), we have

$$\frac{1}{T}a'(t) = J\mathfrak{H}_{zz}(t)a(t). \tag{5.11}$$

Let us seek a semiclassical asymptotic solution to the system (5.11) as

$$a(t) = \mathrm{e}^{\mathrm{i}(T\Phi(s)+\phi(s))}f(t), \tag{5.12}$$

$$f(t) = f^{(0)}(s) + \frac{1}{T}f^{(1)}(s). \tag{5.13}$$

Substituting (5.12) in (5.11) and equating the terms of the same order in $1/T$, we obtain

$$\begin{aligned} &\Phi'(s) = \Omega(s), \qquad \phi'(s) = -\frac{\mu(s)}{2\Omega(s)}\left(\frac{\rho(s)}{\mu(s)}\right)', \\ &f^{(0)}(s) = \frac{1}{\sqrt{\Omega(s)\mu(s)}}\begin{pmatrix} -\rho(s) + \mathrm{i}\Omega(s) \\ \mu(s) \end{pmatrix}. \end{aligned} \tag{5.14}$$

Let us expand the vector $f^{(1)}(s)$ in the basis vectors $f^{(0)}(s)$ and $f^{(0)*}(s)$ to obtain

$$f^{(1)}(s) = \alpha(s) f^{(0)}(s) + \beta(s) f^{(0)*}(s). \tag{5.15}$$

Then

$$\beta(s) = -\frac{\mu(s)}{4\Omega^2(s)} \left(\frac{\rho(s) - \mathrm{i}\Omega(s)}{\mu(s)} \right)'. \tag{5.16}$$

The function $\alpha(s)$ is to be determined from the condition of existence of the next-order approximation. We do not give the explicit form of this approximation, as it does not contribute to the terms of the order of $1/T$.

The matrix $Q(t) = B(t)C^{-1}(t)$ accurate to the first order in $1/T$ reads

$$Q(t) = Q^{(0)}(s) + \frac{1}{T} Q^{(1)}(s) = \frac{-\rho(s) + \mathrm{i}\Omega(s)}{\mu(s)} + \frac{1}{T} \frac{-1}{2\Omega(s)} \left(\frac{\Omega(s) + \mathrm{i}\rho(s)}{\mu(s)} \right)'. \tag{5.17}$$

The function

$$\begin{aligned} \Phi_0^{(0)}(x, t, \mathfrak{C}) = {} & \sqrt[4]{\frac{1}{\pi\hbar}} \left(\frac{1}{|C(t)|} \right)^{1/2} \exp\left[\frac{\mathrm{i}}{2} (\operatorname{Arg} C(0) - \operatorname{Arg} C(t)) \right] \\ & \times \exp\left[\frac{\mathrm{i}}{2\hbar} \left(-\int_0^t \tilde{\varkappa} c(s) \sigma_{xx}(t, \mathfrak{C}, \hbar)\, \mathrm{d}t + Q(t) \Delta x^2 \right) \right] \end{aligned} \tag{5.18}$$

is a solution of the associated equation (3.3) accurate to the first order in $1/T$. Here $\sigma_{xx}(t, \mathfrak{C}, \hbar)$ is determined from expressions (5.8) and (5.9), and $Q(t)$ is given by formula (5.17). For the Fock basis, we obtain the following representation:

$$\begin{aligned} & \Phi_n^{(0)}(x, t, \mathfrak{C}) = \frac{1}{\sqrt{n!}} \Phi_0^{(0)} \frac{1}{\sqrt{2^n}} \exp[-\mathrm{i}n(\operatorname{Arg} C(t) - \operatorname{Arg} C(0))] H_n(\xi), \\ & \xi = \sqrt{\frac{\operatorname{Im} Q^{(0)}(s)}{\hbar}} \Delta x. \end{aligned} \tag{5.19}$$

In view of (5.17), we define $\overline{\mathfrak{C}}_n$ as

$$\overline{\mathfrak{C}}_n = \left(\hbar \frac{\mu(2n+1)}{2\Omega}, 0 \right)^{\mathsf{T}}. \tag{5.20}$$

Accordingly, a solution to equation (5.2) is determined as

$$\Psi_n^{(0)}(x, t) = \Phi_n^{(0)}(x, t, \overline{\mathfrak{C}}_n). \tag{5.21}$$

Note that if (5.3) is taken as an initial condition for finding a solution to equation (5.2), then

$$\Psi(x, t) = \Psi_n^{(0)}(x, t) + O\left(\frac{1}{T} \right), \tag{5.22}$$

where

$$\Psi_n^{(0)}(x, t) = \exp\left\{ -\frac{\mathrm{i}}{\hbar} T \int_0^s E_n(\tau)\, \mathrm{d}\tau + \mathrm{i}\gamma_n(s) \right\} \psi_n(x, R(s)). \tag{5.23}$$

The function (5.23) is a solution of equation (5.2) in the adiabatic approximation. The quantities

$$E_n(s) = \hbar \left(n + \frac{1}{2} \right) \left(\frac{\tilde{\varkappa} c(s)}{2 \operatorname{Im} Q^{(0)}(s)} + \Omega(s) \right) = \hbar \left(n + \frac{1}{2} \right) \left(\frac{\tilde{\varkappa} c(s) \mu(s)}{2\Omega(s)} + \Omega(s) \right) \tag{5.24}$$

are the eigenvalues of the instantaneous Hamiltonian $\hat{\mathcal{H}}_x(R(s), \psi_n(R(s))$, and $\gamma_n(s)$ has the form

$$\begin{aligned}\gamma_n(s) &= -\left(n+\frac{1}{2}\right)\int_0^s \left[\phi'(\tau) - \frac{\tilde{\varkappa}c(\tau)}{2}\frac{\operatorname{Im}Q^{(1)}(\tau)}{(\operatorname{Im}Q^{(0)}(\tau))^2}\right]\mathrm{d}\tau \\ &= -\left(n+\frac{1}{2}\right)\int_0^s \left[1 - \frac{\tilde{\varkappa}c(\tau)}{2}\frac{\mu(\tau)}{\Omega^2(\tau)}\right]\phi'(\tau)\,\mathrm{d}\tau \\ &= \left(n+\frac{1}{2}\right)\int_0^s \left[1 - \frac{\tilde{\varkappa}c(\tau)}{2}\frac{\mu(\tau)}{\Omega^2(\tau)}\right]\frac{\mu(\tau)}{2\Omega(\tau)}\left(\frac{\rho(\tau)}{\mu(\tau)}\right)'\,\mathrm{d}\tau, \\ \Omega(\tau) &= \sqrt{[\sigma(\tau)+\tilde{\varkappa}a(\tau)]\mu(\tau) - \rho^2(\tau)}. \end{aligned} \tag{5.25}$$

Considering the evolution during a period, we obtain

$$\Psi_n^{(0)}(x,T) = \exp\left\{-\frac{\mathrm{i}}{\hbar}T\int_0^1 E_n(s)\,\mathrm{d}s + \mathrm{i}\gamma_n(T)\right\}\Psi_n^{(0)}(x,0). \tag{5.26}$$

Using (1.8) and (1.9), we determine the dynamic phase

$$\delta_n(T) = \left(n+\frac{1}{2}\right)T\int_0^1\left(\frac{\tilde{\varkappa}c(s)\mu(s)}{2\Omega(s)} + \Omega(s)\right)\mathrm{d}s \tag{5.27}$$

and the Berry phase

$$\gamma_n(T) = \left(n+\frac{1}{2}\right)\oint_C\left[1-\frac{\tilde{\varkappa}c}{2}\frac{\mu}{\Omega^2(s)}\right]\frac{1}{2\Omega}\left(\mathrm{d}\rho - \frac{\rho}{\mu}\,\mathrm{d}\mu\right) = \oint_C A_\mu^n\,\mathrm{d}\mu + A_\rho^n\,\mathrm{d}\rho. \tag{5.28}$$

The components of the 'potential' A^n in the parameter space are determined by the following relations:

$$\begin{aligned} A_\mu^n &= -\left(n+\frac{1}{2}\right)\left[1-\frac{\tilde{\varkappa}c}{2}\frac{\mu}{\Omega^2}\right]\frac{1}{2\Omega}\frac{\rho}{\mu}, \\ A_\rho^n &= \left(n+\frac{1}{2}\right)\left[1-\frac{\tilde{\varkappa}c}{2}\frac{\mu}{\Omega^2}\right]\frac{1}{2\Omega}, \\ A_\sigma^n &= A_a^n = A_b^n = A_c^n = 0. \end{aligned} \tag{5.29}$$

The Berry phase can be presented in the form of an integral over a surface Π in the space of parameters, supported by the contour C:

$$\begin{aligned}\gamma_n(T) = \left(n+\frac{1}{2}\right)&\int\int_\Pi \frac{1}{4\Omega^3}\{\bar{\sigma}(\mathrm{d}\rho\wedge\mathrm{d}\mu) + \rho(\mathrm{d}\mu\wedge\mathrm{d}\bar{\sigma}) + \mu(\mathrm{d}\bar{\sigma}\wedge\mathrm{d}\rho)\} \\ &- \frac{\tilde{\varkappa}c}{2}\frac{3\mu}{4\Omega^5}\{\tilde{\sigma}(\mathrm{d}\rho\wedge\mathrm{d}\mu) + \rho(\mathrm{d}\mu\wedge\mathrm{d}\tilde{\sigma}) + \mu(\mathrm{d}\tilde{\sigma}\wedge\mathrm{d}\rho)\}, \\ \tilde{\sigma} = \sigma + \tilde{\varkappa}a, &\qquad \bar{\sigma} = \sigma + \tilde{\varkappa}(a+c). \end{aligned} \tag{5.30}$$

Expression (5.23) confirms the statement of the adiabatic theorem for a nonlinear equation with the Hamiltonian (1.2) in the class of trajectory-concentrated functions.

The difference between the Berry phases for the nonlinear equation (1.1) and for the linear Schrödinger equation ($\tilde{\varkappa} = 0$) is the variation of the frequency Ω and in the appearing additional summand

$$-\left(n+\frac{1}{2}\right)\oint_C\frac{\tilde{\varkappa}c}{2}\frac{\mu}{\Omega^2}\frac{1}{2\Omega}\left(\mathrm{d}\rho - \frac{\rho}{\mu}\,\mathrm{d}\mu\right). \tag{5.31}$$

Note that the calculation of the Berry phase from the states (4.17) by formula (1.11) does not give the term (5.31). Accordingly, expression (5.29) for the 'potentials' differs from that obtained by formula (1.12) by terms proportional to $\tilde{\varkappa}c$. This is so because the states (4.17)

do not satisfy the superposition principle. In the limiting case as $\tilde{\varkappa} \to 0$, the Berry phase coincides with that obtained earlier in [29] (see also [24]).

The classical analogue to a Berry phase is known as a Hannay angle (see e.g. [30]), a component in the dynamic part of the 'angle' variable that arises in an adiabatically evolvable integrable dynamical system described in terms of the 'action-angle' variables. Hannay's angle has a nature similar to that of Berry's phase, and the relationship between Berry's phase γ of a quantum system and Hannay's angle Θ of the corresponding classical system is given by the formula

$$\Theta = -\hbar \frac{\partial \gamma}{\partial I} = -\frac{\partial \gamma}{\partial n}, \tag{5.32}$$

where I is the quantized action and n are quantum numbers. The differentiation with respect to n is performed as if it would be a continuous parameter. In a nonlinear case, it is natural to associate the Hamiltonian $\hat{\mathcal{H}}_\varkappa$ with an analogue of Hannay's angle by formula (5.32)

$$\Theta_\varkappa = -\oint_C \left[1 - \frac{\tilde{\varkappa} c}{2} \frac{\mu}{\Omega^2}\right] \frac{1}{2\Omega} \left(\mathrm{d}\rho - \frac{\rho \, \mathrm{d}\mu}{\mu}\right). \tag{5.33}$$

In the limit $\tilde{\varkappa} \to 0$, we obtain the well-known Hannay angle for the generalized harmonic oscillator [30].

6. Conclusions

In this paper, we have constructed a solution for the one-dimensional nonstationary Gross–Pitaevskii equation (1.1) and have found explicit expressions for the adiabatic Berry phase. The eigenfunctions of the instantaneous nonlinear Hamiltonian (1.2) have been constructed by a semiclassical method based on the Maslov complex germ theory [34, 35]. This method, approximate in general, gives exact solutions for the spectral problem in the case under consideration.

A classical analogue of the Berry phase is the Hannay angle [30]. We have defined the Hannay angle in terms of quantum mechanics, since the nonlinear problem requires a special study of the 'classical equations' corresponding to the nonlinear 'quantum' Gross–Pitaevskii equation. In our consideration, the role of these classical equations is played by the Hamilton–Ehrenfest system (2.2), which has no Hamiltonian form relative to the standard Poisson bracket.

The geometric potentials obtained (5.29), which determine the Berry phase, are Abelian. In the linear quantum-mechanical case ($\varkappa = 0$ in equation (1.1)), they can be treated as effective potentials of an electromagnetic field. Such a situation takes place, e.g., in the theory of the molecular Aharonov–Bohm effect in the Born–Oppenheimer adiabatic approximation. The corrections arising due to this effect are experimentally established in chemical physics [6]. These potentials were also considered in [31].

The non-Abelian Berry phase is of interest in quantum computations [32, 33]. Such a phase appears for an instantaneous Hamiltonian having a degenerate spectrum, which arises in a multidimensional case. The approach developed in this paper admits multidimensional generalization and the results obtained can be used to study the relevant problems.

In conclusion, let us discuss the applicability of the adiabatic approximation $T \to \infty$ to the Gross–Pitaevskii equation (1.1). The nonlinear Schrödinger equation is well known to be used in quantum mechanics for approximate description of linear many-body systems [36]. Special features of the dynamics of systems of this type were studied, in particular, by Jona-Lasinio [37] who revealed numerically that in the thermodynamic limit, the solutions of nonlinear mean-field equations that describe a spinless boson system show chaoticity.

In the BEC theory, a Gross–Pitaevskii equation is obtained from kinetic equations by neglecting the non-coherent counterparts of non-condensate states while taking into account pairwise elastic collisions. Only the coherent evolution of the BEC states is considered. In this case, the Gross–Pitaevskii equation describes the evolution of the BEC for physically significant times, namely, as long as the BEC exists [8, 9]. Physically, a BEC is a metastable state with long lifetimes. In particular, in these limits, the long adiabatic time T can be used. We have constructed asymptotic solutions to equation (1.1) for cases where the adiabatic evolution time can be considered long so that the fractional variations of the adiabatic parameters be small ($\ll 1$). Taken in this sense, the variation of the asymptotic parameter $T \to \infty$ is within the applicability limits of the Gross–Pitaevskii equation (1.1). It should be mentioned that for the scattering problem, solutions asymptotic in $T \to \infty$ were constructed for some special cases of equation (1.1) (see e.g. [38, 39] and references therein).

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References

[1] Messia A 1981 *Quantum Mechanics* vol 2 (Amsterdam: North-Holland)
[2] Berry M V 1984 Quantum phase factors accompanying algebraic changes *Proc. R. Soc.* A **392** 45–58
[3] Vinitskii S I, Derbov V L, Dubovik V M, Markovskii B L and Stepanovskii Yu P 1990 Topological phases in quantum mechanics and polarized optics *Usp. Fiz. Nauk Russ. Acad. Sci.* **160** 1–49 (in Russian)
[4] Moore D J 1991 The calculation of nonadiabatic Berry phase *Phys. Rep.* **210** 1–43
[5] Klyshko D N 1993 Geometric phase in oscillation processes *Usp. Fiz. Nauk Russ. Acad. Sci.* **163** 1–18 (in Russian)
[6] Biswas S N and Soni S K 1990 Quantal phase of Berry and its relation to Hannay's angle in classical mechanics *Proc. Indian Natl Sci. Acad.* A **57** 1–44
[7] Garrison J C and Raymond C Y 1988 Geometrical phases from global gauge invariance of nonlinear classical field theories *Phys. Rev. Lett.* **60** 165–8
[8] Pitaevskii L P 1988 Bose–Einstein condensation in magnetic traps. Introduction to the theory *Usp. Fiz. Nauk* **168** 641–53
Pitaevskii L P 1988 *Phys. Usp.* **4** 1569–80 (Engl. Transl.)
[9] Dalfovo F, Giorgini S, Pitaevskii L P and Stringari S 1999 Theory of Bose–Einstein condensation in trapped gases *Rev. Mod. Phys.* **71** 463–542
[10] Chen Z-D, Liang J-Q, Shen S-Q and Xie W F 2004 Dynamics and Berry phase of two-species Bose–Einstein condensates *Preprint* cond-mat/0403570
[11] Fuentes-Guridi I, Pachos J, Bose S, Vedral V and Choi S 2002 Geometric phases of mesoscopic spin in Bose–Einstein condensates *Phys. Rev.* A **66** 022102
[12] Pitaevskii L P 1961 Vortex lines in an imperfect Bose gas *Zh. Eksp. Teor. Fiz.* **40** 646–51 (in Russian)
[13] Gross E P 1961 Structure of a quantized vortex in boson systems *Nuovo Cimento* **20** 454–77
[14] Zakharov V E, Manakov S V, Novikov S P and Pitaevsky L P 1980 *Theory of Solitons: The Inverse Scattering Method* (Moscow: Nauka)
Zakharov V E, Manakov S V, Novikov S P and Pitaevsky L P 1984 *Theory of Solitons: The Inverse Scattering Method* (New York: Consultants Bureau) (Engl. Transl.)
[15] Kivshar Y S and Malomed B A 1990 Dynamics of solitons in nearly integrable systems *Rev. Mod. Phys.* **8** 763–915
[16] Bang O, Krolikowski W, Wyller J and Rasmussen J J 2002 Collapse arrest and soliton stabilization in nonlocal nonlinear media *Preprint* nlin.PS/0201036
[17] Shchesnovich V S and Kraenkel R A 2004 Vortices in nonlocal Gross–Pitaevskii equation *J. Phys. A: Math. Gen.* **37** 6633–51

[18] Deconinck B and Kutz J N 2002 Singular instability of exact stationary solutions of the nonlocal Gross–Pitaevskii equation *Preprint* cond-mat/0208441
[19] Lisok A L, Trifonov A Yu and Sapovalov A V 2004 The evolution operator of the Hartree-type equation with a quadratic potential *J. Phys. A: Math. Gen.* **37** 4535–56
[20] Lisok A L, Trifonov A Yu and Sapovalov A V 2005 Symmetry operators of a Hartree-type equation with a quadratic potential *Sib. Mat. Zh.* **46** 149–65
Lisok A L, Trifonov A Yu and Sapovalov A V 2005 *Sib. Math. J.* **46** 119–32
[21] Belov V V, Trifonov A Yu and Shapovalov A V 2002 The trajectory-coherent approximation and the system of moments for the Hartree type equation *Int. J. Math. Math. Sci.* **32** 325–70
[22] Belov V V, Trifonov A Yu and Shapovalov A V 2002 Semiclassical trajectory-coherent approximations to Hartree type equations *Teor. Mat. Fiz.* **130** 460–92
Belov V V, Trifonov A Yu and Shapovalov A V 2002 *Theor. Math. Phys.* **130** 391–418 (Engl. Transl.)
[23] Landau L D and Lifshits E M 1986 *Mechanics* (Oxford: Pergamon)
[24] Dodonov V V, Klimov A B and Man'ko V I 1991 Physical phenomena in quantum correlated states *Tr. Fiz. Inst. Lebedev* **200** 56–105
[25] Trifonov A Yu and Yevseyevich A A 1994 Maslov's complex germ method and Berry's phase *J. Phys. A: Math. Gen.* **27** 6267–86
[26] Robertson H P 1934 An indeterminacy relation for several observables and its classical interpretation *Phys. Rev.* **46** 794–801
[27] Belov V V and Kondratieva M F 1994 Hamiltonian systems of equations for quantum means *Mat. Zametki* **56** 27–39
Belov V V and Kondratieva M F 1994 *Math. Notes* **56** 1228–36 (Engl. Transl.)
[28] Yakubovich V A and Starzhinskii V M 1975 *Linear Differential Equations with Periodic Coefficients* (Malabar, FL: Krieger)
[29] Berry M V 1985 Classical adiabatic angles and quantal adiabatic phase *J. Phys. A: Math. Gen.* **18** 15–27
[30] Hannay J H 1985 Angle variable holonomy in adiabatic excursion of an integrable Hamiltonian *J. Phys. A: Math. Gen.* **18** 221–30
[31] Aharonov Y and Stern A 1992 Origin of the geometric forces accompanying Berry's geometric potentials *Phys. Rev. Lett.* **69** 3593–97
[32] Zanardi P and Rasetti M 1999 Holonomic quantum computation *Phys. Lett.* A **264** 94–9
[33] Biswas A J, Vedral V, Ekert A and Castagnol G 2000 Geometric quantum computation using nuclear magnetic resonance *Nature* **403** 869–71
[34] Maslov V P 1977 *The Complex WKB Method in Nonlinear Equations* (Moscow: Nauka)
Maslov V P 1994 *The Complex WKB Method for Nonlinear Equations: I. Linear Theory* (Basle: Birkhauser) (Engl. Transl.)
[35] Belov V V and Dobrokhotov S Yu 1992 Semiclassical Maslov asymptotics with complex phases: I. General approach *Teor. Mat. Fiz.* **92** 215–54
Belov V V and Dobrokhotov S Yu 1992 *Theor. Math. Phys.* **92** 843–68 (Engl. Transl.)
[36] Hartree D R 1928 The wave mechanics of an atom with a non-Coulomb central field: I *Proc. Camb. Phil. Soc.* **24** 89–110
Hartree D R 1928 The wave mechanics of an atom with a non-Coulomb central field: II *Proc. Camb. Philos. Soc.* **24** 111–32
Hartree D R 1928 The wave mechanics of an atom with a non-Coulomb central field: III *Proc. Camb. Philos. Soc.* **24** 426–37
[37] Jona-Lasinio G and Presilla C 1996 Chaotic properties of quantum many-body systems in the thermodynamic limit *Phys. Rev. Lett.* **77** 4322–5
[38] Hayashi N and Naumkin P I 1998 Remarks on scattering theory and large time asymptotics of solutions to Hartree type equations with a long range potential *SUT J. Math.* **34** 13–24
[39] Ginibre J and Velo G 2000 Long range scattering and modified wave operators for some Hartree type equations. I *Rev. Math. Phys.* **12** 361–429
Ginibre J and Velo G 2000 Long range scattering and modified wave operators for some Hartree type equations. II *Ann. Inst. Henri Poincaré* **1** 753–800